

USING ELASTIC-PLASTIC PLANE STRESS FEM-PROGRAMS TO COMPUTE ANTI-PLANE STRAIN

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Summary

Elastic-plastic anti-plane strain problems can be solved by using FEM-programs developed for plane stress. The technique does not involve any manipulations with the FEM-code, but is merely a matter of specialization and of translation of denotations. Two analytically solvable cases are chosen for demonstration.

Introduction

Hardly any of the standard FEM-codes is supplied with routines for elastic-plastic anti-plane strain analysis. Many codes do not even provide routines for the simpler linear case (involving the Poisson equation, which is the governing equation for a variety of problems: torsion, steady state heat conduction, electrostatic and magnetostatic potential fields, just to mention a few).

One obvious, but extremely inefficient way to treat anti-plane strain problems would be to use three-dimensional elements. The method presented here offers a better alternative, in which two-dimensional plane stress elements are used. Thereby much higher efficiency is obtained and a larger number of different elements is generally available than at a three-dimensional treatment.

As a by-product of the technique a possibility is opened to check the plane stress code itself for the elastic-plastic case, since more non-trivial analytical solutions are available for anti-plane strain than for plane stress.

Plane stress-anti-plane strain analogy

The elastic-plastic case.

The governing equations for a strain hardening elastic-plastic material under plane stress conditions can be written (with obvious denotations):

$$\begin{cases} \epsilon_x = \partial u / \partial x = \epsilon_x^e + \epsilon_x^p \end{cases} \quad (1)$$

$$\begin{cases} \epsilon_y = \partial v / \partial y = \epsilon_y^e + \epsilon_y^p \end{cases} \quad (2)$$

$$\gamma_{xy} = \partial v / \partial x + \partial u / \partial y = \gamma_{xy}^e + \gamma_{xy}^p \quad (3)$$

$$\begin{cases} d\epsilon_x^e = 1/E (d\sigma_x - \nu d\sigma_y) \end{cases} \quad (4)$$

$$\begin{cases} d\epsilon_y^e = 1/E (d\sigma_y - \nu d\sigma_x) \end{cases} \quad (5)$$

$$\begin{cases} d\gamma_{xy}^e = 2(1+\nu)/E d\tau_{xy} \end{cases} \quad (6)$$

$$\begin{cases} d\epsilon_x^p = (\sigma_x - 1/2\sigma_y) d\lambda \end{cases} \quad (7)$$

$$\begin{cases} d\epsilon_y^p = (\sigma_y - 1/2\sigma_x) d\lambda \end{cases} \quad (8)$$

$$\begin{cases} d\gamma_{xy}^p = 3 \tau_{xy} d\lambda \end{cases} \quad (9)$$

where

$$d\lambda = \begin{cases} d\epsilon^p / \sigma_e & \text{if } \sigma_e = \sigma_Y \text{ and } d\sigma_e = 0 \\ 0 & \text{if } \sigma_e < \sigma_Y \text{ or if } \sigma_e = \sigma_Y \text{ and } d\sigma_e < 0 \end{cases} \quad (10)$$

for the hardening rate $H = 0$ and

$$d\lambda = \begin{cases} d\sigma_e / (\sigma_e H) & \text{if } \sigma_e = \sigma_f \text{ and } d\sigma_e > 0 \\ 0 & \text{if } \sigma_e < \sigma_f \text{ or if } \sigma_e = \sigma_f \text{ and } d\sigma_e \leq 0 \end{cases} \quad (11)$$

for $H \neq 0$,

$$d\epsilon^p = 2/3^{1/2} [(d\epsilon_x^p)^2 + (d\epsilon_y^p)^2 + d\epsilon_x^p d\epsilon_y^p + 1/4 (d\gamma_{xy}^p)^2]^{1/2} \quad (12)$$

$$\sigma_e = (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2)^{1/2} \quad (13)$$

σ_f is the flow stress, which equals the yield stress σ_Y at infinitesimal small plastic strain. The hardening rate H may be constant (linear hardening) or a function of the effective plastic strain, $\epsilon^p = \int d\epsilon^p$. The equations of equilibrium are

$$\begin{cases} \partial \sigma_x / \partial x + \partial \tau_{xy} / \partial y = K_x \end{cases} \quad (14)$$

$$\begin{cases} \partial \sigma_y / \partial y + \partial \tau_{xy} / \partial x = K_y \end{cases} \quad (15)$$

where K_x and K_y are the body forces per unit volume.

For anti-plane strain the governing equations read (with obvious denotations):

$$\begin{cases} \gamma_{xz} = \partial w / \partial x = \gamma_{xz}^e + \gamma_{xz}^p \end{cases} \quad (16)$$

$$\begin{cases} \gamma_{yz} = \partial w / \partial y = \gamma_{yz}^e + \gamma_{yz}^p \end{cases} \quad (17)$$

$$\begin{cases} d\gamma_{xz}^e = 1/G \, d\tau_{xz} \end{cases} \quad (18)$$

$$\begin{cases} d\gamma_{yz}^e = 1/G \, d\tau_{yz} \end{cases} \quad (19)$$

$$\begin{cases} d\gamma_{xz}^p = 3\tau_{xz} \, d\lambda \end{cases} \quad (20)$$

$$\begin{cases} d\gamma_{yz}^p = 3\tau_{yz} \, d\lambda \end{cases} \quad (21)$$

where

$$d\lambda = \begin{cases} d\varepsilon^p / \sigma_e & \text{if } \sigma_e = \sigma_Y \text{ and } d\sigma_e = 0 \\ 0 & \text{if } \sigma_e < \sigma_Y \text{ or if } \sigma_e = \sigma_Y \text{ and } d\sigma_e < 0 \end{cases} \quad (22)$$

for $H = 0$ and

$$d\lambda = \begin{cases} d\sigma_e / (\sigma_e H) & \text{if } \sigma_e = \sigma_f \text{ and } d\sigma_e > 0 \\ 0 & \text{if } \sigma_e < \sigma_f \text{ or if } \sigma_e = \sigma_f \text{ and } d\sigma_e \leq 0 \end{cases} \quad (23)$$

for $H \neq 0$.

$$d\varepsilon^p = (1/3)^{1/2} [(d\gamma_{xz}^p)^2 + (d\gamma_{yz}^p)^2]^{1/2} \quad (24)$$

and

$$\sigma_e = 3^{1/2} (\tau_{xz}^2 + \tau_{yz}^2)^{1/2} \quad (25)$$

The equation of equilibrium is

$$\partial \tau_{xz} / \partial x + \partial \tau_{yz} / \partial y = K_z \quad (26)$$

where K_z is the body force per unit volume.

Now by suitable specializations and changes of denotations the equation system (1)-(14) for plane stress can be transformed to the system (16)-(26) for anti-plane strain. To achieve this it is necessary first to put $\nu = 1/2$ and then assume that $K_y(x, y)$ in (15) is so chosen that $\nu = 0$ throughout. Then Equation (2) gives



$$d\epsilon_y^e + d\epsilon_y^p = 0 \quad (27)$$

and thus, by using Equations (5) and (8)

$$\sigma_y = \sigma_x/2 \quad (28)$$

By introducing $\nu = 1/2$, $\nu = 0$ and $\sigma_y = \sigma_x/2$ in Equations (1), (3), (4), (6), (7), (9)-(14) and then making the following changes in denotations:

$$\begin{aligned} E/3 &\rightarrow G \\ x/2 &\rightarrow x \\ u &\rightarrow w \\ 2\epsilon_x &\rightarrow \gamma_{xz} \\ \gamma_{xy} &\rightarrow \gamma_{yz} \\ 2\epsilon_x^e &\rightarrow \gamma_{xz}^e \\ \gamma_{xy}^e &\rightarrow \gamma_{yz}^e \\ 2\epsilon_x^p &\rightarrow \gamma_{xz}^p \\ \gamma_{xy}^p &\rightarrow \gamma_{yz}^p \\ \sigma_x/2 &\rightarrow \tau_{xz} \\ \tau_{xy} &\rightarrow \tau_{yz} \\ K_x &\rightarrow K_z \end{aligned} \quad (29)$$

one arrives at Equations (16)-(26). Furthermore Equations (2), (5) and (8) do not contain any of the quantities appearing in Equations (16)-(26) and can therefore be left aside. The remaining Equation (15), which is now written

$$\partial\tau_{xz}/\partial y + 1/2 \partial\tau_{yz}/\partial x = K_y \quad (30)$$

simply defines a now uninteresting quantity K_y .

The linearly elastic case.

The governing equations for a linearly elastic material subjected to anti-plane loading conditions read

$$\gamma_{xz} = \partial w / \partial x = 1/G \tau_{xz} \quad (31)$$

$$\gamma_{yz} = \partial w / \partial y = 1/G \tau_{yz} \quad (32)$$

The equation of equilibrium is identical to (26). In this case an analogy between plane stress and anti-plane strain is possible without reference to any specific value of ν . Equations (26), (31) and (32) are obtained by putting $\nu = 0$ in Equations (1), (3), (4), (6) and (14) after making the following changes in denotations:

$$\begin{aligned} [2(1+\nu)]^{-1} E &\rightarrow G \\ [(1-\nu)/2]^{1/2} \epsilon_x &\rightarrow x \\ u &\rightarrow w \\ [2/(1-\nu)]^{1/2} \epsilon_x &\rightarrow \gamma_{xz} \\ \gamma_{xy} &\rightarrow \gamma_{yz} \\ [(1-\nu)/2]^{1/2} \sigma_x &\rightarrow \tau_{xz} \\ \tau_{xy} &\rightarrow \tau_{yz} \\ K_x &\rightarrow K_z \end{aligned} \quad (33)$$

Even though a specific value of ν is not necessary, the choice $\nu = -1+\omega$, where $\omega \ll 1$, is very convenient, because this implies practically seen unchanged geometry. (For numerical reasons the ideal choice $\nu = -1$ cannot be used).

By using Equations (31) and (32), Equation (26) gives

$$\partial^2 w / \partial x^2 + \partial^2 w / \partial y^2 = K_z / G \quad (34)$$

which is the Poisson equation for the linear elastic anti-plane strain problem. Application to other problems governed by the Poisson equation can be made by appropriate interpretation of the quantities w and K_z/G in Equation (34). Thus, for the Poisson equation

$$(\partial^2 U / \partial x^2 + \partial^2 U / \partial y^2) = S \quad (35)$$

the following changes in denotations are made in Equations (1), (3), (4), (6) and (14) after putting $v = 0$, $\nu = -1+\omega$ and (without loss of universality) $E = 2\omega$:

$$\begin{aligned} (1-\omega/2)^{1/2} x &\rightarrow x \\ u &\rightarrow U \\ (1-\omega/2)^{1/2} \sigma_x &\rightarrow \partial U / \partial x \\ \tau_{xy} &\rightarrow \partial U / \partial y \\ K_x &\rightarrow S \end{aligned} \quad (36)$$

For the special case $\omega \ll 1$ one can approximate $(1-\omega/2)^{1/2} \approx 1$. This is also demonstrated in an example at the end of this paper.

The translation of boundary conditions.

As usual when finite element methods are applied, distributed loads and boundary tractions are represented by suitably chosen concentrated forces.

Figure 1 shows the constant cross-section of a very long bar subjected to concentrated loads Z_i per unit length, perpendicular to the cross-section and uniformly distributed along the lines $x = x_i$, $y = y_i$. Anti-plane strain is assumed, i.e. the displacements $u = 0$ and $v = 0$ throughout the body. The boundary conditions are assumed to consist of prescribed displacements $w_0(x,y)$ in the z -direction on AB and of prescribed tractions $T_z = 0$ on the remaining part of the boundary. Then, for the elastic-plastic case, according to Equation (29) the corresponding plane stress problem concerns a thin plate with essentially the shape of the cross-section in Figure 1 but expanded to double size in the x direction (see Figure 2). ν is put equal to zero throughout the plate. Concentrated forces $X_i = 2Z_i$ per unit length, acting in the positive x -direction, are applied at $x = 2x_i$, $y = y_i$. The boundary conditions are prescribed displacements $u(x,y) = w_0(x/2,y)$ on the arc A'B' corresponding to AB and the tractions, acting in the x -direction, $T_x = 0$ on the remaining part of the boundary. Furthermore $E = 3G$ and $\nu = 1/2$ whereas σ_y and H remain unchanged. The anti-plane displacements are found from the plane stress solution as $w(x,y) = u(2x,y)$.

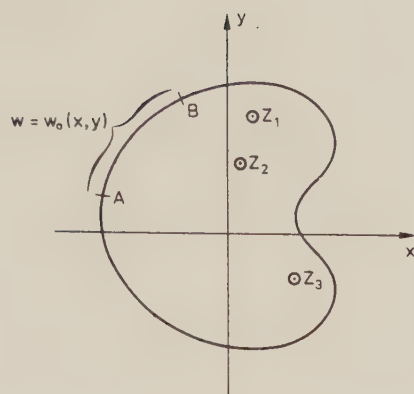


Figure 1. The anti-plane strain problem studied. Z_i are concentrated loads and $w_0(x, y)$ displacements along AB.

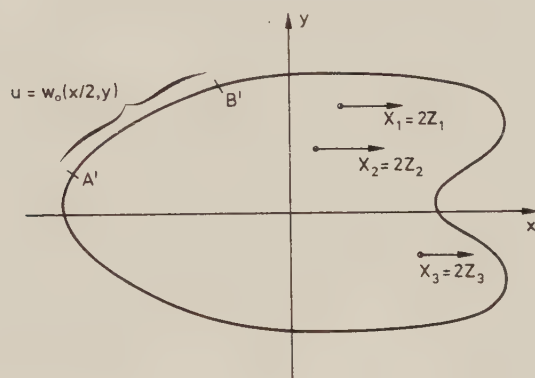


Figure 2. Boundaries, boundary conditions and loads for the plane stress problem analogous to the anti-plane strain problem shown in Figure 1.

For the linearly elastic case with the choice $\nu = -1 + \omega$, where $\omega \ll 1$, an identical geometry (see Figure 3) can be assumed for a plane stress problem, i.e. $x \rightarrow x$. Further $X_i = Z_i$ on $x = x_i$, $y = y_i$, $u(x, y) = w_0(x, y)$ on AB, $T_x = 0$ on the remaining part of the boundary, $\nu = 0$ throughout and $E = 2\omega G$. The anti-plane displacements are found from the plane stress solution as $w(x, y) = u(x, y)$.

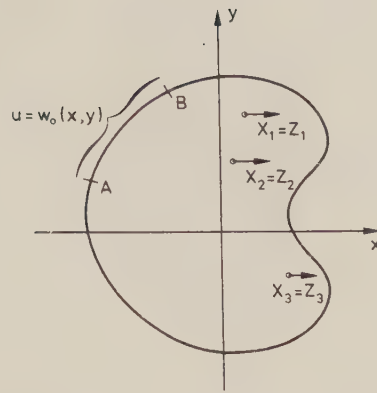


Figure 3. Boundaries, boundary conditions and loads for a plane stress problem analogous to a linear anti-plane strain problem with boundary conditions and loads as in Figure 1.

A mode III elastic-plastic crack problem

In order to illustrate the procedure in the elastic-plastic case an analytically solvable problem will be studied. Consider an infinitely long elastic-perfectly-plastic cylinder with elliptic cross-section, $4x^2 + y^2 = 4a^2$. A sharp crack is situated in the x - z -plane with the crack edge at $x = y = 0$ (see Figure 4). Small scale yielding is assumed. The boundary traction T_z , in the z -direction, is taken as the dominating term of the Williams expansion [1] around the point $x = (3/2\pi)(K_{III}/\sigma_Y)^2$, $y = 0$ (where K_{III} is the stress intensity factor and σ_Y is the yield stress). After some calculations, using also a polar coordinate system r , θ with the origin at the crack tip and $\theta = \pm\pi$ on the crack surfaces, one obtains:

$$T_z = K_{III} [2\pi r_*]^{-1/2} [\sin\theta \cos(\theta_*/2) - 4\cos\theta \sin(\theta_*/2) / (15\cos^2\theta + 1)^{1/2}] \quad (37)$$

$$\text{on } 4x^2 + y^2 = 4a^2$$

$$\text{where } r_* = [x_*^2 + y^2]^{1/2}, \quad \theta_* = \tan^{-1}(y/x_*),$$

$$\text{and } x_* = x - 3/(2\pi)(K_{III}/\sigma_Y)^2$$

The mesh for the corresponding plane stress problem is shown in Figure 5.

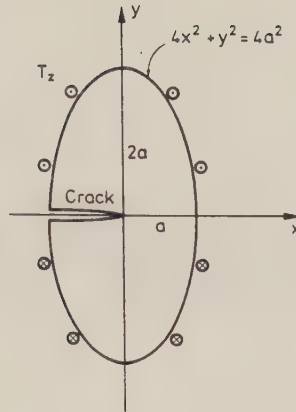


Figure 4. Infinite elastic-plastic cylinder. Anti-plane strain. T_z is the load on the boundary.

Due to the symmetry only the upper half, $y \geq 0$, of the cylinder needs to be studied. The mesh consists of 64 8-node isoparametric elements and the element stiffnesses are taken from 2 by 2 integration points. The elements at the crack tip are quadrilaterals degenerated to triangles as one side is collapsed to form a single point at the crack tip. The nodes at the crack tip are treated as separate nodes in the analysis, irrespective of the fact that they coincide as long as the body is unloaded. Hence the resulting displacement at the crack tip is depending on the angle θ from which $r = 0$ is approached:

$$u \rightarrow f(\theta) \quad \text{as} \quad r \rightarrow 0 \quad (38)$$

so

$$\gamma_{z\theta} = \partial u / \partial \theta \cdot r^{-1} \rightarrow f'(\theta) r^{-1} \quad \text{as} \quad r \rightarrow 0 \quad (39)$$

i.e. an r^{-1} -type of strain singularity is allowed (cf [2]), which is exactly what is expected for this kind of problems.

According to Equation (29) the boundary traction T_x , in the x-direction, for the plane stress problem is

$$T_x = K_{III} [2\pi r_*]^{-1/2} [\sin\theta \cos(\theta_*/2) - 4\cos\theta \sin(\theta_*/2)] / (3\cos^2\theta + 1)^{1/2} \quad (40)$$

$$\text{on} \quad x^2 + y^2 = 4a^2$$

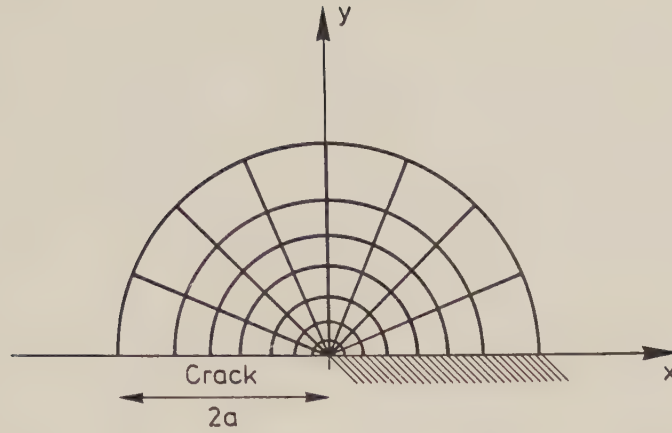


Figure 5. Principal element mesh for the plane stress problem analogous to the anti-plane strain problem shown in Figure 4.

where $r_* = [x_*^2/4 + y^2]^{1/2}$, $\theta = \tan^{-1}(2y/x)$, $\theta_* = \tan^{-1}(2y/x_*)$,

$$x_* = x - (3/\pi)(K_{III}/\sigma_Y)^2$$

and the displacement

$$u = 0 \quad \text{on} \quad x > 0, \quad y = 0 \quad (41)$$

The solution for monotonic loading is characterized by self-similarity since the stresses depend only on r through the combination

$$r\sigma_Y^2/K_{III}^2 \quad (42)$$

This self-similarity is exploited by a numerical procedure operating according to [4]. The final load (40) is applied in one step and iterations are performed to obtain equilibrium. At final load, which is chosen to $K_{III} = \sigma_Y(0.8a)^{1/2}$, the plastic zone is extended to approximately the distance $0.8a$ ahead of the crack tip.

The result is interpreted for anti-plane strain through Equation (29) and compared with the analytical result [3] as regards displacements along the crack surface, displacements along a path surrounding the crack tip and displacements for different directions θ as $r \rightarrow 0$ (see Figures 6 and 7).

The displacements along the crack surface are found to coincide with the analytical solution with a maximum error of 0.2 %. The error along a path surrounding the crack tip is larger, probably as a consequence of the strain singularity, but the agreement with the analytical solution is obvious. Figure 8 shows the plastic zone and the stress vector $\bar{\tau} = \tau_{yz}\hat{x} - \tau_{xz}\hat{y}$. According to the analytical solution this vector, at any point inside the plastic zone,

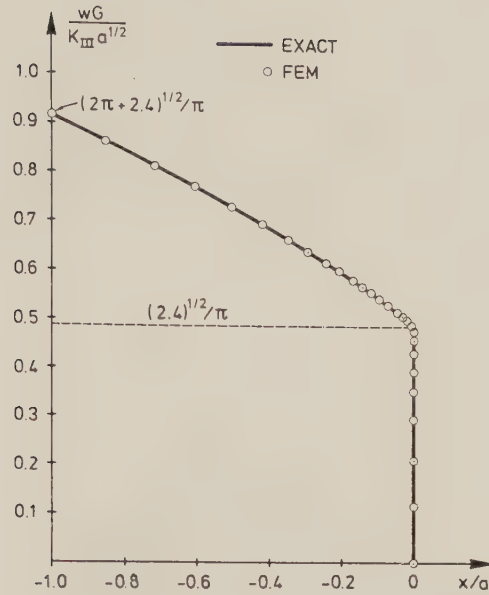


Figure 6. Non-dimensional displacement $wG/K_{III}a^{1/2}$ versus x/a along the upper crack surface, $y = 0$. At $x = 0$, i.e. at the crack tip, the displacement depends on the angle from which $x = 0$ is approached (see Figure 7)

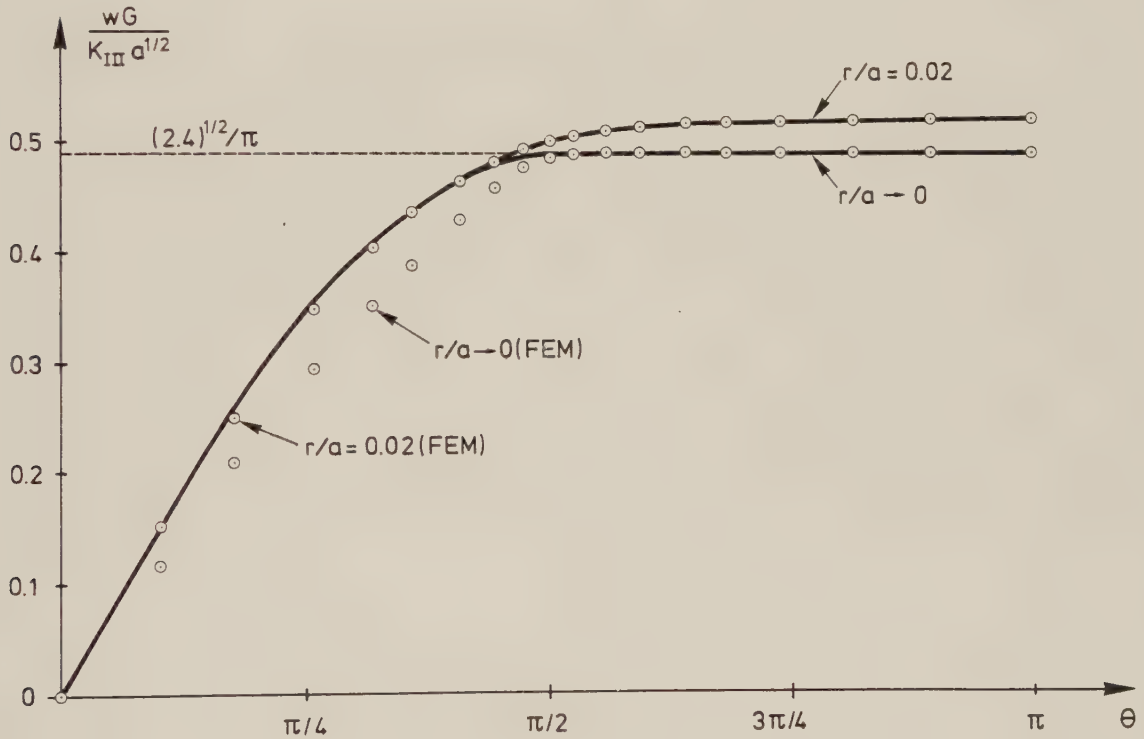


Figure 7. Non-dimensional displacement $wG/K_{III}a^{1/2}$ versus the angle θ from the positive x -axis for the non-dimensional distances from the crack tip $r/a = 0.02$ and $r/a \rightarrow 0$.

intersects the x-axis at the crack tip. This is not quite obtained by the numerical calculations, which result in intersections between $\bar{\tau}$ and the x-axis slightly behind the crack tip. The intersection point is about 1/2 of an element length (i.e. about 1/40 of the plastic zone size) behind the crack tip.

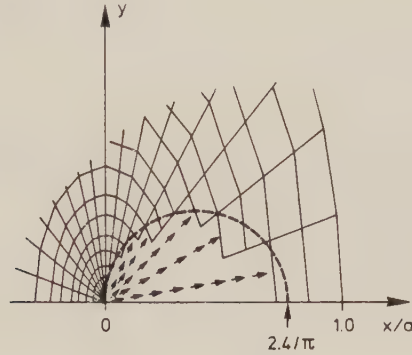


Figure 8. The shape of the plastic zone and the direction of the stress vector $\bar{\tau} = \tau_{yz}x - \tau_{xz}y$ (shown as arrows), for $K_{III} = \sigma_Y(0.8a)^{1/2}$.

A problem on steady state heat transfer

As previously mentioned other interpretations than those given by Equation (33) are appropriate for other problems governed by the Poisson equation than linearly elastic anti-plane strain problems. Steady state heat conduction is one example. As an illustration the continuous heat transfer between two pipes embedded in an infinite heat-conducting medium is considered (see Figure 9). Pipe A has radius r_0 and pipe B has radius $2r_0$. The distance between the centres of the two pipes is $7r_0$. The pipes are assumed to be infinitely long, thus invoking plane conditions. The temperature of pipe A is put equal to zero and the temperature of pipe B is T_0 . Radiation is assumed to

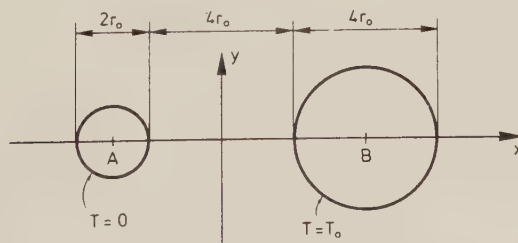


Figure 9. Two pipes with the temperatures $T = 0$ and $T = T_0$ respectively, embedded in an infinite medium.

be negligible. (Since no heat sources are acting in the medium, $S = 0$ in Equation (35), i.e. the Poisson equation is degenerated to the Laplace equation in this case.)

The following changes of denotations in the plane stress problem are made, c.f. Equation (36):

$$\begin{aligned} (1-\omega/2)^{1/2}x &\rightarrow x \\ u &\rightarrow T \end{aligned} \quad (43)$$

where T is the temperature. ω is chosen as 0.01, alternatively as 0.001. Due to these choices of ω , $x \rightarrow 0.995^{-1/2}x$, alternatively $x \rightarrow 0.9995^{-1/2}x$. Here, however, this is approximated with $x \rightarrow x$ for both cases.

The analytical solution is easily found by using bipolar coordinates,

$$\begin{cases} x = 4/7 \cdot \sqrt{30} \cdot \sinh\eta / (\cosh\eta - \cos\xi) \\ y = 4/7 \cdot \sqrt{30} \cdot \sin\xi / (\cosh\eta - \cos\xi) \end{cases} \quad (44)$$

giving

$$T = T_0 \ln \left\{ \frac{4\sqrt{30}-7x}{4\sqrt{30}+7x} \cdot \frac{\sqrt{30}-4}{\sqrt{30}+4} \right\} / \ln(11-2\sqrt{30}) \quad \text{for } y = 0 \quad (45)$$

Figure 10 shows the FEM results T_{FEM} compared with the analytical result T_{AN} . The results for $\omega = 0.01$ and $\omega = 0.001$ coincide within the resolution of

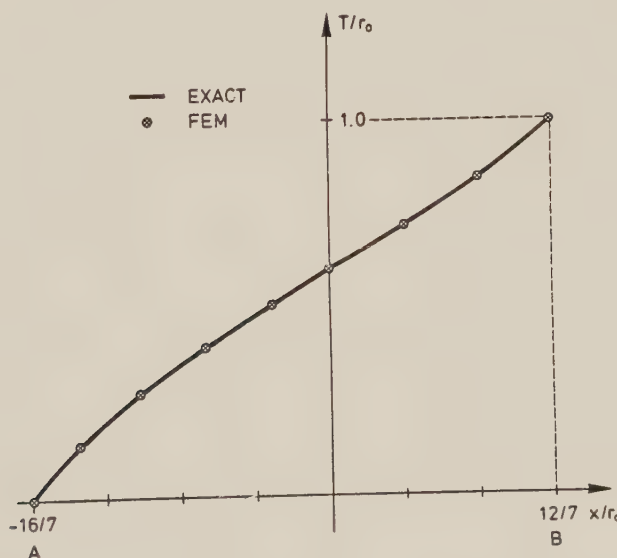


Figure 10. The temperature distribution along the x -axis, between the pipes. The results from the FEM-calculations and the exact result coincide within the resolution of the diagram. The maximum error is less than 0.2% of the temperature difference T_0 between the pipes.

the diagram. The maximum error $|T_{AN} - T_{FEM}|/T_0$ is 0.125% for $\omega = 0.01$ and 0.175% for $\omega = 0.001$, indicating that the errors are not introduced by the fact that ω cannot be chosen as 0.

Conclusions

Elastic-plastic anti-plane strain problems can be solved by treating an analogous plane stress problem considering a deformed geometry. For the corresponding linear problem, or other problems governed by the Poisson equation, the original geometry can be retained.

Naturally, the advantage of the method is economical or practical. A code designed for the special purpose, if available, should, of course, be more efficient.

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